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Helmholtz Conditions and Alternative Lagrangians: Study of an Integrable Hénon–Heiles System

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Some properties relating to the theory of constants of motion with the existence of alternative Lagrangians are given, and then the case of the harmonic oscillator is analyzed. The $\epsilon = 1/3$ Hénon–Heiles system is studied as a deformation of the harmonic oscillator. Several different alternative Lagrangians compatible with this deformation are constructed.

1. INTRODUCTION

The search for constants of motion in the Lagrangian approach has been traditionally related to the existence of one-parameter subgroups of point transformations that are symmetries of the Lagrangian, according to the celebrated Noether theorem. However, it has also been proved in recent years that constants of motion can also be found from the knowledge of alternative but non-gauge-equivalent Lagrangians. Hojman and Harleston proved that when there are two alternative regular Lagrangians it is possible to define a related matrix such that the trace of such a matrix as well as those of the sequence of powers of it are constants of motion. We recall that by alternative or "*s*-equivalent" (*s* is for solution) Lagrangians we mean two non-gauge-equivalent Lagrangian functions that lead to different Euler–Lagrange equation systems that, however, admit the same set of solutions.[(1–8)]

The number of known systems admitting alternative Lagrangians is small and therefore they must be considered as exceptional. The existence of alternative Lagrangians seems to be directly related to the theory of nonlinear integrable systems. Moreover, it is connected with some interesting problems such as the existence of nonequivalent quantizations,[(9)] the

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existence and properties of dynamical symmetries,[6] or the theory of Lax equations.[10–12] In field theories, this matter is related to soliton equations.

Using the symplectic formalism, a regular Lagrangian function defines a symplectic form $\omega_L = -d\theta_L$ and an energy function $E_L = \Delta(L) - L$ on the velocity phase space TQ (θ_L is the associated Cartan one-form and Δ the Liouville vector field) in such a way that the dynamics is given by the vector field Γ_L solution of the equation $i(\Gamma_L)\omega_L = dE_L$. Because of this, the existence of two alternative Lagrangians $L^{(a)}$, $a = 1, 2$, for a certain system Γ_L means the existence of two different symplectic structures $\omega_2 \neq \omega_1$ and two different energy functions $E_2 \neq E_1$ for the same dynamics on the same manifold. Consequently, the theory of alternative Lagrangians must be considered as a particular case of the more general theory of bi-Hamiltonian systems.

In this paper, we study the existence of alternative Lagrangians for one of the three integrable cases of the Hénon–Heiles system,[13–19] using as an approach the ideas arising from the so-called theorem of Hojman and Harleston. We first study in Section 2 the relation between the existence of alternative Lagrangians and the properties of constants of motion and then in Section 3 we obtain several alternative Lagrangians for the harmonic oscillator. In Section 4, which contains our main results, we study the Hénon–Heiles system as a deformation of the harmonic oscillator and we obtain four different Lagrangians for the $\epsilon = 1/3$ Hénon–Heiles system, all of them compatible with the deformation. In Section 5 we make some final comments.

2. THE INVERSE PROBLEM AND THE EXISTENCE OF ALTERNATIVE LAGRANGIANS

The inverse problem of the Lagrangian dynamics studies the conditions for a system of second-order differential equations

$$\ddot{q}^i = f^i(q, \dot{q}) \tag{1}$$

to be the Euler–Lagrange equations of some regular Lagrangian L. This equation may be reduced to the first-order system

$$\frac{dq^i}{dt} = v^i$$

$$\frac{dv^i}{dt} = f^i(q, v) \tag{2}$$

From the geometric viewpoint, the velocity phase space is the tangent bundle $\tau\colon TQ \to Q$ of the configuration space Q and the preceding system can be considered as the one determining the curves projecting on the base space of the local integral curves of the second-order vector field $\Gamma \in \mathfrak{X}(TQ)$ with local coordinate expression

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i(q, v) \frac{\partial}{\partial v^i}$$

The inverse problem can be presented in a more generalized sense: Are there functions $g_{ij}(q, \dot{q})$ with $\det[g_{ij}] \neq 0$ and a regular Lagrangian L such that $g_{ij}(\ddot{q}^j - f^j) = 0$ are the Euler–Lagrange equation system of the function L? The problem was studied Helmholtz,[20] who established the so-called Helmholtz conditions (see refs. 21–23). In terms of the functions f^i, these conditions are given as follows: There should exist a family of functions $g_{ij} = g_{ij}(q, v)$ such that

(i) $g_{ij} = g_{ji}$

(ii) $\det[g_{ij}] \neq 0$

(iii) $\dfrac{\partial g_{ij}}{\partial v^k} = \dfrac{\partial g_{ik}}{\partial v^j}$

(iv) $\Gamma(g_{ij}) = \dfrac{1}{2} g_{kj} \dfrac{\partial f^k}{\partial v^i} + \dfrac{1}{2} g_{ik} \dfrac{\partial f^k}{\partial v^j}$

(v)
$$g_{ik}\left[\frac{\partial f^k}{\partial q^j} + \frac{1}{4}\frac{\partial f^k}{\partial v^l}\frac{\partial f^l}{\partial v^j} - \frac{1}{2}\Gamma\left(\frac{\partial f^k}{\partial v^j}\right)\right] = g_{jk}\left[\frac{\partial f^k}{\partial q^i} + \frac{1}{4}\frac{\partial f^k}{\partial v^l}\frac{\partial f^l}{\partial v^i} - \frac{1}{2}\Gamma\left(\frac{\partial f^k}{\partial v^i}\right)\right]$$

In the affirmative case these five properties lead to the existence of a function L such that the g_{ij} take the form

$$g_{ij} = \frac{\partial^2 L}{\partial v^i \partial v^j}$$

The regularity of L is consequence of condition (ii). In geometric terms these conditions are related to the existence of symplectic structures.[22–25]

Hojman and Harleston proved,[1] generalizing a result of Currie and Saletan[26] for the $n = 1$ case, that if a system admits two alternative regular Lagrangians $L^{(a)}$, $a = 1, 2$, then the traces of the powers of the product matrix $W_{21} = W_2^{-1} W_1$ are constants of motion (W_a denotes the Hessian matrix of $L^{(a)}$). If these constants of motion are nontrivial it is said that they have been obtained by a non-Noether procedure.

Consequently, the existence of alternative Lagrangians for a certain system leads to the existence of integrals of motion and hence is related to the theory of integrable systems. Moreover, Crampin *et al.*[4] proved that if the (1, 1) tensor field R defined by $\omega_2(X, Y) = \omega_1(RX, Y)$ satisfies some properties, then the n integrals obtained by Hojman and Harleston are in involution and the system is therefore completely integrable. Conversely, only systems endowed with a certain number of constants of motion, as, e.g., integrable or superintegrable systems, can admit alternative Lagrangians.

The number of known systems admitting alternative Lagrangians is very small and usually they correspond to the case of velocity-independent forces. The above theorem of Hojman and Harleston can be considered as a non-Noether procedure for obtaining integrals of motion, but when the system has velocity-free forces (see, e.g., ref. 7) then this procedure becomes simpler. In this case the following two properties are satisfied:

1. The functions $W_{ij} = \partial^2 L/\partial v^i \partial v^j$, $i, j = 1, \ldots, n$, are constants of motion.
2. The functions $B_{ij} = \partial^2 L/\partial v^i \partial q^j$, $i, j = 1, \ldots, n$, are symmetric.

The important point is that when a system Γ_L admits alternative regular Lagrangians $L^{(a)} \neq L$, $a = 1, \ldots, A$, then properties 1 and 2 are also true for every one of these alternative Lagrangians. This property agrees with the Helmholtz approach since if the functions f^i are velocity-independent, then the three last conditions reduce to the following:

(iiib) $\dfrac{\partial g_{ij}}{\partial v^k} = \dfrac{\partial g_{ik}}{\partial v^j}$

(ivb) $\Gamma(g_{ij}) = 0$

(vb) $g_{ik}\left(\dfrac{\partial f^k}{\partial q^j}\right) = g_{jk}\left(\dfrac{\partial f^k}{\partial q^i}\right)$

Next we consider the one- and the two-dimensional cases.

If $n = 1$, then $g_{ij}(q, \dot{q})$ reduce to a unique function $g(q, \dot{q})$. In this case four of the five Helmholtz conditions become identities and the only one to be satisfied just states that g must be a constant of motion, $dg/dt \equiv \Gamma(g) = 0$. This g can be taken as a real number $g \equiv 1$, so all the one-dimensional systems are Lagrangian [this is a well-known result since $f(q)$ can always be written as the derivative of another function which can be denoted as $-V(q)$]. The important point is that if the system admits a Lagrangian L, then every function $g(E_L)$ of the associated energy E_L is a constant of motion. Consequently, one-dimensional Lagrangian systems always admit an infinite number of alternative Lagrangians[26] (one of them is, of course, the standard one).

Suppose now the $n = 2$ case. Then we can state[7] following proposition:

Proposition 1. Let Γ be a $n = 2$ degrees of freedom dynamical system $\Gamma = v^i\partial/\partial q^i + f^i\partial/\partial v^i$, with $f^i = f^i(q^1, q^2)$, $i = 1, 2$. Suppose that $F_r = F_r(q, v)$, $r = 1, 2, 3$, are three functions such that:

(a) The functions $F_r(q, v)$, $r = 1, 2, 3$, are constants of motion; i.e., $\Gamma(F_r) = 0$.
(b) $\partial F_2/\partial v^1 = \partial F_1/\partial v^2$ and $\partial F_2/\partial v^2 = \partial F_3/\partial v^1$.
(c) The three functions F_r and the two forces f^i satisfy the matrix equation

$$\begin{pmatrix} F_1 & F_2 \\ F_2 & F_3 \end{pmatrix} \begin{pmatrix} \dfrac{\partial f^1}{\partial q^1} & \dfrac{\partial f^1}{\partial q^2} \\ \dfrac{\partial f^2}{\partial q^1} & \dfrac{\partial f^2}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f^1}{\partial q^1} & \dfrac{\partial f^1}{\partial q^2} \\ \dfrac{\partial f^2}{\partial q^1} & \dfrac{\partial f^2}{\partial q^2} \end{pmatrix} \begin{pmatrix} F_1 & F_2 \\ F_2 & F_3 \end{pmatrix}$$

Then:

1. The vector field Γ is a dynamical system arising from a regular Langrangian L, i.e., $\Gamma = \Gamma_L$.
2. The three functions $F_r(q, v)$, $r = 1, 2, 3$, are the matrix elements, $F_1 = W_{11}$, $F_2 = W_{21}$, $F_3 = W_{22}$, of the Hessian W of the Lagrangian L.

Notice that the three functions F_r can be functionally dependent (even trivial) provided that the matrix is nonsingular. The important point is that every set of such three functions determines a particular Lagrangian. So, if for a certain Γ we can construct several appropriate sets of functions $F_r^{(a)}$, $a = 1, \ldots, A$, then Γ will be a Lagrangian system admitting several different s-equivalent Lagrangians $L^{(a)}$, $a = 1, \ldots, A$.

3. ALTERNATIVE LAGRANGIANS FOR THE HARMONIC OSCILLATOR

Let us first consider an $n = 2$ system with Lagrangian L representing a direct sum of two noninteracting one-dimensional systems

$$L = \frac{1}{2}\,(v_x^2 + v_y^2) - U(x) - W(y)$$

This system is separable, the energy of each degree of freedom being a constant of motion. A simple way of obtaining alternative Lagrangians to take into account that L can be written in the form $L = L_x + L_y$ with

$$L_x = \frac{1}{2}\,v_x^2 - U(x), \qquad L_y = \frac{1}{2}\,v_y^2 - W(y)$$

considering equivalent Lagrangians in one dimension, L'_x and L'_y, and defining $L' = L'_x + L'_y$.

For studying a more general Lagrangian mixing both degrees of freedom, we see that condition (c) leads to

$$F_2(U''_{xx} - W''_{yy}) = 0$$

Hence if $F_2 \neq 0$, we obtain

$$U = \frac{1}{2}Ax^2 + c_1 x, \qquad W = \frac{1}{2}Ay^2 + c_2 y$$

Since the linear terms can be eliminated by a translation, we arrive at the isotropic oscillator (a related result can be found in ref. 3).

1. The standard Lagrangian of the two-dimensional harmonic oscillator can be considered as a particular case of the function[6]

$$L^{(1)} = \frac{a}{2}(v_x^2 - Ax^2) + \frac{d}{2}(v_y^2 - Ay^2) + b(v_x v_y - Axy)$$

where a, b, d are real constants such that $ad - b^2 \neq 0$. In this case $W_{11}^{(1)} = a$, $W_{12}^{(1)} = b$, and $W_{22}^{(1)} = d$, which are trivial constants of motion.

The function $L^{(1)}$ can be considered as rather close to the standard Lagrangian, nevertheless there exist other, not so simple alternative Lagrangians. Next we will prove that these more general alternative functions can be explicitly obtained by integration; the first step of the method is the construction of a matrix satisfying the necessary conditions to be considered as a Hessian matrix.

The isotropic harmonic oscillator is a superintegrable system[27–30] with three independent constants: $I_1 = (1/2)(v_x^2 + Ax^2)$, $I_2 = (1/2)\,(v_y^2 + Ay^2)$, and $I_3 = yv_x - xv_y$. Nevertheless, instead of working with I_3, we will make use of the nondiagonal component of the Fradkin tensor $I_4 = v_x v_y + Axy$.[31] In this way the three constants will be quadratic in the velocities. Hence, if we restrict the study to Hessian functions W_{ij}, $i, j = 1, 2$, linear in the three integrals I_1, I_2, and I_4, then the associated Lagrangians will be functions of fourth order in the velocities.

We omit the details of the computations and give directly the regular Lagrangians obtained. We only make two remarks. First, in this case the matrix of partial derivatives of the forces is a multiple of the identity matrix; therefore condition (c) is trivially satisfied. Second, in the following we will always assume that $F_2 \neq 0$.

2. Let us consider as functions F_r, $r = 1, 2, 3$, the following set of constants of motion: $F_1 = I_2$, $F_2 = I_4$, and $F_3 = I_1$. We have $\partial F_2/\partial v_x = \partial F_1/\partial v_y = v_y$ and $\partial F_2/\partial v_y = \partial F_3/\partial v_x = v_x$. Thus we know that there must exist a new Lagrangian $L^{(2)}$ such that its Hessian must take the form $W_{11}^{(2)} =$

I_2, $W_{12}^{(2)} = I_4$, and $W_{22}^{(2)} = I_1$. After integration we obtain that this $L^{(2)}$ is given by

$$L^{(2)} = \frac{1}{4}(v_x^2 v_y^2) + \frac{1}{4}k(y^2 v_x^2 + x^2 v_y^2) + kxyv_x v_y - \frac{3}{4}k^2 x^2 y^2$$

3. If we take $W_{11}^{(3)} = I_4$, $W_{12}^{(3)} = I_1$, and $W_{22}^{(3)} = 0$, then

$$L^{(3)} = \frac{1}{6}v_x^3 v_y + \frac{1}{2}A(xyv_x^2 + x^2 v_x v_y) - \frac{1}{2}A^2 x^3 y$$

4. If we take $W_{11}^{(4)} = 0$, $W_{12}^{(4)} = I_2$, and $W_{22}^{(4)} = I_4$, then

$$L^{(4)} = \frac{1}{6}v_x v_y^3 + \frac{1}{2}A(xyv_y^2 + y^2 v_x v_y) - \frac{1}{2}A^2 xy^3$$

Although these three functions $L^{(a)}$, $a = 2, 3, 4$, can be considered as the three fundamental Lagrangians of degree four in the velocities, we will see that the following two alternative Lagrangians, to be denoted by $L^{(5)}$ and $L^{(6)}$, will be more closely related to the Hénon–Heiles system.

5. Now let us consider $F_1 = I_1 + I_2$, $F_2 = I_4$, and $F_3 = I_1 + I_2$. They satisfy the three conditions of Proposition 1. Therefore $W_{11}^{(5)} = I_1 + I_2$, $W_{12}^{(5)} = I_4$, and $W_{22}^{(5)} = I_1 + I_2$, from which we obtain

$$L^{(5)} = \frac{1}{24}(v_x^4 + v_y^4) + \frac{1}{4}v_x^2 v_y^2 + \frac{1}{4}A(x^2 + y^2)(v_x^2 + v_y^2) + Axyv_x v_y - \frac{1}{8}A^2(x^4 + 6x^2 y^2 + y^4)$$

6. If we take $W_{11}^{(6)} = I_4$, $W_{12}^{(6)} = I_1 + I_2$, and $W_{22}^{(6)} = I_4$, then

$$L^{(6)} = \frac{1}{6}(v_x^3 v_y + v_x v_y^3) + \frac{1}{2}Axy(v_x^2 + v_y^2) + \frac{1}{2}A(x^2 + y^2)v_x v_y - \frac{1}{2}A^2(x^3 y + xy^3)$$

All these new Lagrangians $L^{(a)}$ determine different equations, but the same dynamics as the standard Lagrangian L. The only problem is that the function $\det W^{(a)}$ $(a \neq 1)$ is not constant on the phase space $TQ \approx \mathbb{R}^4$; so every $L^{(a)}$ will define not only the dynamics of the harmonic oscillator, but also another one of singular character on the submanifold $M^{(a)} \subset \mathbb{R}^4$ which is the zero-level set of the function $\det W^{(a)}$. For example, for $L^{(3)}$ and $L^{(4)}$ we obtain

$M^{(3)} = I_1^{-1}(0)$ and $M^{(4)} = I_2^{-1}(0)$; concerning $L^{(5)}$ and $L^{(6)}$, we obtain

$$\det W^{(5)} = -\det W^{(6)} = (I_1 + I_2)^2 - I_4^2$$
$$= [A(y - x)^2 + (v_y - v_x)^2]\,[A(y + x)^2 + (v_y + v_x)^2]$$

and hence

$$M^{(5)} = M^{(6)} = N_1 \cup N_2$$

where

$$N_1 = \{(x, y, v_x, v_y)\big| y = x,\ v_y = v_x\}$$

and

$$N_2 = \{(x, y, v_x, v_y)\big| y = -x,\ v_y = -v_x\}$$

The existence of alternative Lagrangians is not limited to $L^{(1)}$ and all these fourth-order functions. As a last alternative Lagrangian we will derive a function of sixth order in the velocities.

7. If we take $W_{11}^{(7)} = (I_1 + I_2)^2 + I_4^2$, $W_{12}^{(7)} = 2(I_1 + I_2)I_4$, and $W_{22}^{(7)} = (I_1 + I_2)^2 + I_4^2$, then these three functions satisfy property (b) and they lead to

$$L^{(7)} = \frac{1}{120}(v_x^6 + v_y^6) + \frac{1}{8}(v_x^4 v_y^2 + v_x^2 v_y^4) + \frac{A}{24}[(x^2 + y^2)(v_x^4 + 6v_x^2 v_y^2 + v_y^4)$$
$$+ 8Axy(v_x^3 v_y + v_x v^3)_y] + \frac{A^2}{8}[(x^4 + 6x^2y^2 + y^4)(v_x^2 + v_y^2)$$
$$+ 8(x^3y + xy^3)v_x v_y] - \frac{A^3}{24}(x^6 + 15x^4y^2 + 15x^2y^4 + y^6)$$

Before studying the Hénon–Heiles system, we remark that a Lagrangian L determines an associated energy function E_L and that for most of Lagrangians the energy E_L coincides with the Newtonian energy $E_N = T + V$. If two Lagrangians are gauge equivalent, then they determine the same energy, that is, if $L^{(2)} = L^{(1)} + dg/dt$, then $E_L^{(2)} = E_L^{(1)}$. This property is not usually true for the case of s-equivalence. Actually, every one of the alternative Lagrangians obtained in this section determines an energy $E_L^{(a)} \neq T + V$ (even in the case of $L^{(1)}$ with $a \neq 1$, $b \neq 0$, and $d \neq 1$). The point is that the Helmholtz conditions guarantee the existence of a Lagrangian, but they do not impose that this Lagrangian be of the form $T(q, v) - V(q)$ with the kinetic term T quadratic in velocities. However, although $E_L^{(a)} \neq E_L^{(b)}$, $a \neq b$, all of them are constants of motion, that is,

$$\frac{d}{dt}E_L^{(a)} \equiv \Gamma_L(E_L^{(a)}) = 0, \qquad a = 1, 2, \ldots, A$$

Similarly, the flow of infinitesimal symmetries preserves the set of integral curves of the system (in our case the harmonic oscillator) and these symmetries very often appear as symmetry of the Lagrangian (in some cases the symmetry is hidden as for the Runge–Lenz vector). When the Lagrangian is not uniquely determined, an infinitesimal symmetry appears in a different way in each Lagrangian. Although we are not concerned in this paper with quantum mechanics, this is a question that seems to be related to the existence of different ways of quantizing the classical system, here the harmonic oscillator.

4. ALTERNATIVE LAGRANGIANS FOR THE $\epsilon = 1/3$ HÉNON–HEILES SYSTEM

The standard Lagrangian $\mathbb{L}$ of the Hénon–Heiles system takes the form

$$\mathbb{L} = \frac{1}{2}(v_x^2 + v_y^2) - \frac{1}{2}(Ax^2 + By^2) - x^2y - \epsilon y^3$$

Only three integrable cases are known[13–18]: (i) $\epsilon = 1/3$, $B = A$, (ii) $\epsilon = 2$, A and B arbitrary, and (iii) $\epsilon = 16/3$, $B = 16A$. Since $n = 2$, integrability means just the existence of a second constant of motion J_2 (J_1 is the energy). The two first cases (i) and (ii) are Hamilton–Jacobi-separable with J_2 quadratic in the velocities. The separability of case (iii) has been recently studied making use of non-point transformations.[18,19]

The Hénon–Heiles system can be considered as the result of a "deformation" of the harmonic oscillator. By "deformation" we mean a new system (or family of systems) depending on a certain number of parameters λ_k, $k = 1, \ldots, K$, in such a way that when $\lambda_k \to 0$ we recover the original undeformed system. Concerning the Lagrangian of the $\epsilon = 1/3$ case, it arises as the $\lambda = 1$ particular value of the λ-dependent Lagrangian $\mathbb{L}$ given by

$$\mathbb{L}_\lambda = \frac{1}{2}(v_x^2 + v_y^2) - \frac{1}{2}(Ax^2 + Ay^2) - \lambda\left[x^2y + \frac{1}{3}y^3\right]$$

This more general function $\mathbb{L}_\lambda$ determines an integrable system for arbitrary values of the parameter λ; the corresponding constant of motion is given by

$$J_2(\lambda) = v_xv_y + Axy + \lambda\left[\frac{1}{3}x^3 + xy^2\right]$$

Consequently, these two functions, the Lagrangian $\mathbb{L}_\lambda$ and constant $J_2(\lambda)$, satisfy

$$\lim_{\lambda\to 0}\mathbb{L} = L, \qquad \lim_{\lambda\to 0}J_2(\lambda) = I_4$$

We remark that the $\epsilon = 1/3$ case is an integrable deformation of a superintegrable system. That is, the original undeformed system is endowed with three integrals of motion (i.e., Lagrangian L with three symmetries) and the $\epsilon = 1/3$ Hénon–Heiles system with only two. Therefore, the deformation preserves integrability, but the new system has fewer symmetries than the original one.

In order to look for new λ-dependent Lagrangians $\mathbb{L}^{(a)}$ alternative to the standard function $\mathbb{L}$ we will follow the same strategy as for obtaining the functions $L^{(a)}$, $a = 1, \ldots, 8$, for the harmonic oscillator; that is, make use of the properties (a)–(c).

We begin with point (c). For the general case of the Hénon–Heiles problem it takes the form

$$\begin{pmatrix} A + 2y & 2x \\ 2x & B + 6\epsilon y \end{pmatrix}\begin{pmatrix} W_{11} & W_{12} \\ W_{12} & W_{12} \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{12} & W_{22} \end{pmatrix}\begin{pmatrix} A + 2y & 2x \\ 2x & B + 6\epsilon y \end{pmatrix}$$

Hence we obtain

$$2x(W_{11} - W_{22}) + (B + 6\epsilon y - A - 2y)W_{12} = 0$$

Consequently, if $A = B$ and $\epsilon = 1/3$, then this equation leads to $W_{22} = W_{11}$ with W_{12} arbitrary.

We have obtained the following four Lagrangians.

1. If we take $W_{11}^{(1)} = W_{22}^{(1)} = 0$, $W_{12}^{(1)} = 1$, then the Lagrangian $\mathbb{L}^{(1)}$ takes the form

$$\mathbb{L}^{(1)} = v_x v_y - Axy - \lambda\left[\frac{1}{3}x^3 + xy^2\right]$$

Besides being a new Lagrangian for the $\epsilon = 1/3$ system, it satisfies

$$\lim_{\lambda\to 0} \mathbb{L}^{(1)} = v_x v_y - Axy$$

Hence, the function $L^{(1)}$ is compatible with the deformation for $a = d = 0$. Moreover, as $\mathbb{L}^{(1)}$ is a quadratic function, it is directly related to the constant J_2. The following proposition shows this interesting characteristic.

Proposition 2. The constant of motion J_2 of the $\epsilon = 1/3$ Hénon–Heiles system is the energy $\mathbb{E}^{(1)}$ of the alternative Lagrangian $\mathbb{L}^{(1)}$.

The proof follows directly from the definition of the energy, that is, $\mathbb{E}^{(1)} = \Delta(\mathbb{L}^{(1)}) - \mathbb{L}^{(1)} = J_2$. This property must be considered as a peculiarity of $\mathbb{L}^{(1)}$ and, in a sense, it singles this particular function out from the other alternative Lagrangians.

2. Let us consider $F_1 = F_3 = J_1$ and $F_2 = J_2$. They satisfy the three conditions of proposition 1. Thus there exist a new Lagrangian $\mathbb{L}^{(2)}$ with Hessian matrix $W_{11}^{(2)} = W_{22}^{(2)} = J_1$, $W_{12}^{(2)} = J_2$. It takes the form

$$\mathbb{L}^{(2)} = \frac{1}{24}(v_x^4 + v_y^4) + \frac{1}{4}v_x^2 v_y^2 + \frac{1}{2}\left[\frac{1}{2}A(x^2 + y^2) + \lambda\left(x^2 y + \frac{1}{3}y^3\right)\right](v_x^2 + y_y^2)$$
$$+ \left[Axyv_x v_y + \lambda\left(\frac{1}{3}x^3 + xy^2\right)\right]v_x v_y - \frac{1}{8}A^2(x^4 + 6x^2y^2 + y^4)$$
$$- \frac{1}{6}\lambda A(5x^4 y + 10x^2y^3 + y^5) - \frac{1}{18}\lambda^2(x^6 + 15x^4y^2 + 15x^2y^4 + y^6)$$

3. If we take $W_{11}^{(3)} = W_{22}^{(3)} = J_2$, $W_{12}^{(3)} = J_1$, then we obtain the following function:

$$\mathbb{L}^{(3)} = \frac{1}{6}(v_x^3 v_y + v_x v_y^3) + \frac{1}{2}\left[Axy + \lambda\left(\frac{1}{3}x^3 + xy^2\right)\right](v_x^2 + v_y^2) + \frac{1}{2}\left[A(x^2 + y^2)\right.$$
$$\left.+\lambda\left(x^2 y + \frac{1}{3}y^3\right)\right]v_x v_y - \frac{1}{2}A^2(x^3 y + xy^3) - \frac{1}{6}\lambda A(x^5 + 10x^3y^2 + 5xy^4)$$
$$-\frac{1}{9}\lambda^2(3x^5 y + 10x^3y^3 + 3xy^5)$$

In the following we will denote by F_k and G_k the functions

$$F_k = \frac{1}{2}[(x + y)^k + (x - y)^k], \qquad G_k = \frac{1}{2}[(x + y)^k - (x - y)^k]$$

Making use of this notation, we can rewrite the Lagrangians $\mathbb{L}^{(2)}$ and $\mathbb{L}^{(3)}$ as

$$\mathbb{L}^{(2)} = L^{(5)} + \frac{\lambda}{18}[3G_3(v_x^2 + v_y^2) + 6F_3 v_x v_y - 3AG_5 - \lambda F_6]$$
$$\mathbb{L}^{(3)} = L^{(6)} + \frac{\lambda}{18}[3F_3(v_x^2 + v_y^2) + 6G_3 v_x v_y - 3AF_5 - \lambda G_6]$$

Consequently, these two Lagrangians $\mathbb{L}^{(2)}$ and $\mathbb{L}^{(3)}$ satisfy

$$\lim_{\lambda\to 0}\mathbb{L}^{(2)} = L^{(5)}, \qquad \lim_{\lambda\to 0}\mathbb{L}^{(3)} = L^{(6)}$$

We close this section by obtaining a function $\mathbb{L}^{(4)}$ of sixth order in the velocities.

4. If we take $W_{11}^{(4)} = J_1^2 + J_2^2$, $W_{12}^{(4)} = 2J_1 J_2$, and $W_{22}^{(4)} = J_1^2 + J_2^2$, properties (a) and (b) are satisfied; we have obtained that this new Lagrangian takes the form

$$\mathbb{L}^{(4)} = L^{(7)} + \left(\frac{\lambda}{36}\right)[G_3(v_x^4 + 6v_x^2v_y^2 + v_y^4) + 4F_3(v_x^3v_y + v_xv_y^2)]$$
$$+\left(\frac{\lambda}{18}\right)[(3AG_5 + \lambda F_6)(v_x^2 + v_y^2) + 2(3AF_5 + \lambda G_6)v_xv_y]$$
$$-\left(\frac{\lambda}{9}\right)\left[\left(\frac{\lambda A}{2}\right)F_8 + \left(\frac{3A^2}{4}\right)G_7 + \left(\frac{\lambda^2}{9}\right)G_9\right]$$

It satisfies

$$\lim_{\lambda\to 0} \mathbb{L}^{(4)} = L^{(7)}$$

5. FINAL COMMENTS

We have studied the $\epsilon = 1/3$ Hénon–Heiles system and proven that it admits several alternative Lagrangians. All these new functions can be considered as deformations of alternative Lagrangians previously obtained for the harmonic oscillator. In this study we have made use of two important properties: (i) The second integral of the deformed system ($\lambda \neq 0$) is quadratic, and (ii) the original system ($\lambda = 0$) is superintegrable. Concerning the two other cases, the situation is different: the second case (i.e., $\epsilon = 2$) satisfies (i), but not (ii), and, conversely, the last case (i.e., $\epsilon = 16/3$) satisfies (ii), but not (i).

Finally, another interesting system that can also be considered as a deformation of the harmonic oscillator is the so-called Smorodinsky–Winternitz system.[29] It seems natural to study the Helmholtz conditions for this superintegrable system (in this case the results must be independent of the dimension). We think that these questions should be investigated in the future.

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